Amplitude equations for systems with long-range interactions

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We derive amplitude equations for interface dynamics in pattern forming systems with long-range interactions. The basic condition for the applicability of the method developed here is that the bulk equations are linear and solvable by integral transforms. We arrive at the interface equation via long-wave asymptotics. As an example, we treat the Grinfeld instability, and we also give a result for the Saffman-Taylor instability. It turns out that the long-range interaction survives the long-wave limit and shows up in the final equation as a nonlocal and nonlinear term, a feature that to our knowledge is not shared by any other known long-wave equation. The form of this particular equation will then allow us to draw conclusions regarding the universal dynamics of systems in which nonlocal effects persist at the level of the amplitude description.

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I. INTRODUCTION

A number of pattern formation problems involving an interface are computationally difficult despite the fact that the bulk equations are linear. Examples include Laplacian dynamics (diffusion-limited aggregation, limit of vanishing Péclet number in dendritic growth), flow problems in the Stokes approximation, and elastic problems. They are difficult because of the long-range self-interaction of the interface mediated by the external field. With the availability of modern computer power, these problems can usually be solved to good accuracy in two dimensions. But none of them has been successfully treated on a large scale in three dimensions so far.

Whenever an interface equation can be derived, this offers the advantage of better tractability, both numerically and analytically. Interface equations for three-dimensional systems are two dimensional, rendering much larger systems accessible than bulk simulations [1]. For problems containing a length scale restricting the interaction distance, such as the diffusion length in solidification problems, interface equations are *local* in the long-wave limit [2,3], i.e., they are partial differential equations. The Laplace or Lamé equations, however, do not have a cutoff distance. Hence, it is interesting and important to investigate how this feature affects the long-wave limit, if any. There have been interface equations containing nonlocal but *linear* terms in the literature before [4–6]. As it will turn out, the nonlocal aspects of the system we consider are much more ferocious, leading to terms that are both nonlocal *and* nonlinear. We are also aware of the case of a previously derived amplitude equation containing nonlocal nonlinearities [7,8]. However, the equation in question is weakly nonlinear whereas we will obtain a strongly nonlinear equation.

Let us recall that close to the instability threshold for the merging of order, systems exhibiting a type I_s instability in the nomenclature of Cross and Hohenberg [9], i.e., an instability at finite wave number, are all expected to be described by a universal amplitude equation of the Ginzburg-Landau type. Otherwise, when the critical wave number approaches zero (type II_s) the dynamics is described by a long-wave equation such as the Kuramoto-Sivashinsky (KS) equation. A prerequisite for these derivations is that nonlocal interactions are screened off beyond some length (small in comparison with the inverse of the bifurcation wave number. When long-range interactions (e.g., in electrostatic, magnetic, elastic systems, etc.) are effectively present, no universal equation has been known as yet for the latter case. The results of [7,8] pertain to the former case, i.e., they constitute a generalization of the Ginzburg-Landau description including nonlocal interactions. We derive here a universal form of dynamics in the long-wavelength limit when long-range interactions prevail and obtain a result that is complementary to the Kuramoto-Sivashinsky description. The derivation is exemplified on a system undergoing a surface instability due to elastic stress. Besides the universal feature of the resolved question, we present the practical virtue of this strategy.

Applying our approach to a particular system, a uniaxially strained solid undergoing the Grinfeld instability [10,11], a problem that has recently attracted the community of crystal growth [12–14], we obtain an equation that, when truncated to the leading-order nonlinearities should provide an example of the universal dynamics. As we shall see, this equation contains fewer parameters than to be expected, which is most likely due to the fact that in the case of the Grinfeld instability the dynamics is variational. Introducing an additional free parameter, we are able to give the generic form of the universal equation. A derivation of this equation from symmetry considerations will be presented elsewhere [15].

The paper is organized as follows. In Sec. II, we write down the model equations for the elastically strained system and introduce the appropriate rescaling to perform an asymptotic analysis. Section III describes the asymptotic expansion and matching procedure and presents the final interface equation for two dynamical situations corresponding to conservative and to nonconservative dynamics. In Sec. IV, we give some simulation results for these equations. Section V contains a discussion of universality aspects and gives an example of a simulation of the generic equation. Finally, we will briefly summarize some conclusions and give an outlook in Sec. VI.

II. MODEL EQUATIONS AND NONDIMENSIONALIZATION

The physics of the Grinfeld instability has been discussed in some detail in Ref. [16]. We consider the case of a solid in contact with its melt; let us call the interface position $\zeta(x,t)$ (i.e., we restrict ourselves to two dimensions). It is related to the stress distribution via [12,16–18]:

$$\frac{\zeta_t(x,t)}{\sqrt{1+\zeta_x^2}}=-\frac{1}{k\rho_s}\left(\frac{1-\nu^2}{2E}[(\sigma_{tt}-\sigma_{nn})^2-\sigma_0^2]+\gamma\kappa+\Delta\rho g\zeta\right). \tag{1}$$

Herein, $1/k$ is the mobility, ρ_s the density of the solid, E is Young's modulus, ν the Poisson ratio, γ is the surface tension, $\Delta\rho$ the density contrast between solid and liquid, and g the gravitational acceleration. $\kappa=-\zeta_{xx}/(1+\zeta_x^2)^{3/2}$ is the interface curvature, and partial derivatives of ζ are denoted by subscripts. σ_{tt} and σ_{nn} are tangential and normal stresses at the interface, i.e., $\sigma_{nn}=n_i\sigma_{ij}n_j$, $\sigma_{tt}=t_i\sigma_{ij}t_j$, where n_i and t_i are the components of the normal and the tangent to the interface. By σ_0, we denote the externally imposed uniaxial stress, and the interface position is measured from the equilibrium position of a planar interface. (Setting $\kappa=0$ and $|\sigma_{tt}-\sigma_{nn}|=|\sigma_0|$ in Eq. (1) leads to a steady state $\zeta=0$.) Equation (1) describes the mass current density that is proportional to chemical potential changes produced by strain, capillarity, and gravity.

To solve this equation, it must be supplemented with the equations describing mechanical equilibrium. Assuming linear isotropic elasticity,

$$\sigma_{ij}=\frac{E}{1+\nu}\left(u_{ij}+\frac{\nu}{1-2\nu}u_{kk}\delta_{ij}\right), \tag{2}$$

where u_{ij} is the strain tensor, expressible via the elastic displacements u_i according to $u_{ij}=\frac{1}{2}(\partial_i u_j+\partial_j u_i)$, the condition for mechanical equilibrium, $\partial_j\sigma_{ij}=0$ (neglecting gravity in the bulk as a small cross effect), takes the form of the Lamé equations:

$$(1-2\nu)\nabla^2\mathbf{u}+\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})=0. \tag{3}$$

Note that the static version is sufficient since all interface motions are slow in comparison with sound propagation. Boundary conditions are

$$\sigma_{nn}=\sigma_{nt}=0 \tag{4}$$

at the interface, periodicity in the x direction with some prescribed wavelength $2\pi/q$, and

$$\sigma_{ij}\to\sigma_0\delta_{ix}\delta_{jx} \quad \text{for} \quad z\to-\infty. \tag{5}$$

Herein, $\sigma_{nt}=n_i\sigma_{ij}t_j$ is the shear stress at the interface, which must evidently vanish if the latter is in contact with a fluid. Another cross effect has been neglected here, viz. that of capillary overpressure, leading to a curvature dependent jump of σ_{nn} across the interface. An overall constant pressure, not influencing the dynamics of the interface, has been subtracted out of the definition of σ_{nn}.

Due to the free boundary character, Eqs. (2) through (5) do not completely specify the problem and there is a need for an additional condition, which is just Eq. (1) expressing the fact that the chemical potential difference between solid and liquid is proportional to the normal mass current across the interface. This means that we assume the interface to be microscopically rough (so that a linear relation holds between the current and its conjugate variable, the chemical potential difference) and the solid to be in contact with its liquid or vapor.

Note, however, that for the solution of the elastic problem at one instant of time with given interface position we do not need Eq. (1) (since we have neglected sound propagation effects). This suggests to attack the problem in two stages: first solve the elastic equations, then move the interface according to the elastic fields, and repeat the procedure for the new interface position.

The question is how to obtain a closed interface equation, and to identify its generic form. In order to proceed, we first nondimensionalize the above equations by introducing two length scales $\ell_1=\gamma E/\sigma_0^2(1-\nu^2)$ (the *Griffith* length [19], apart from a factor of $4/\pi$) and $\ell_2=(1-\nu^2)\sigma_0^2/2\Delta\rho g E$ (a gravity length) as well as a time scale $\tau=k\rho_s\ell_1^2/\gamma$ [18]. Then we use Hooke's law (2) to express stresses by strains and reduce the latter by a dimensionless factor $E/(1+\nu)\sigma_0$. We find

$$\frac{\zeta_t(x,t)}{\sqrt{1+\zeta_x^2}}=-\left\{\frac{1}{2}[(u_{tt}-u_{nn})^2-1]+\kappa+\alpha\zeta\right\}, \tag{6}$$

where $\alpha=\ell_1/2\ell_2$, i.e., $\alpha\geqslant 0$. A linear stability analysis of the full set of Eqs. (2) through (6) yields the dispersion relation [18]

$$\omega=2|q|-q^2-\alpha, \tag{7}$$

where ω and q are the nondimensional growth rate and wave number, respectively. The absolute value of q arises, because the eigensolutions of the elastic problem carry a factor $\exp(\pm iqx+|q|z)$, approaching zero for $z\to-\infty$. A linear instability will arise for $\alpha<1$.

As is obvious from the derivation, to be able to manifest itself the instability requires mass transport. Besides the case described of a liquid in contact with its melt, corresponding to nonconserved dynamics, another situation is of particular interest, due to possible applications in epitaxial growth. This is a solid in vacuum. Material transport will ordinarily be dominated by surface diffusion then. Consequently, we have conserved dynamics, with Eq. (6) replaced by

$$\frac{\zeta_t(x,t)}{\sqrt{1+\zeta_x^2}}=\frac{1}{\sqrt{1+\zeta_x^2}}\,\partial_x\,\frac{1}{\sqrt{1+\zeta_x^2}}\times\partial_x\left\{\frac{1}{2}[(u_{tt}-u_{nn})^2-1]+\kappa+\alpha\zeta\right\}, \tag{8}$$

and the dispersion relation (7) is simply multiplied by a factor q^2 on the right-hand side. Since the gravity term is normally negligible in experiments on epitaxial growth (unless a temperature gradient mimics a strong gravitational field [16,17]), we may set $\alpha=0$ in this case, obtaining a parameter-free equation.

As it turns out, the asymptotic analysis can be performed together for both dynamical situations, since its most extensive part consists in solving the elastic problem (which is the same for both cases). Only afterwards is the solution inserted into the equation describing the interface evolution. At intermediate steps of the calculation, we will therefore write down equations for the nonconservative case only, but we shall give the final interface equation for both cases.

Let us now consider a situation, where ℓ_1 is very large, i.e., we introduce a small parameter ϵ, setting $\ell_1=O(1/\epsilon)$. This corresponds to a very small stress, but we can nevertheless reach an unstable state by reducing gravity or the density difference, so ℓ_2 becomes larger than $\ell_1/2$.

Then it will be natural to measure length scales in units of $\epsilon\ell_1$ [which is $O(1)$] rather than ℓ_1 and time scales in units of $\epsilon^2\tau$. Coordinates are transformed according to $\tilde{x}=x/\epsilon$, $\tilde{z}=z/\epsilon$, and $\tilde{t}=t/\epsilon^2$. Referring to the new coordinates, the dispersion relation transforms into $\tilde{\omega}=2\epsilon|\tilde{q}|-\tilde{q}^2-\epsilon^2\alpha$. For small ϵ, each term in this relation must behave as ϵ^2, hence we have for the wave number $\tilde{q}\sim\epsilon$. A long-wave equation should therefore be derivable. To perform the asymptotics, we introduce another set of coordinates via $X=\epsilon\tilde{x}$, $Z=\tilde{z}-\zeta(x,t)$, and $T=\epsilon^2\tilde{t}$, conveniently mapping the interface position to $Z=0$. X and T are slow variables. We set $U=u_x-(1-\nu)X$ and $V=u_z+\nu(Z+\zeta)$, where u_x and u_z are the displacements, which makes the boundary conditions at $-\infty$ homogeneous $(U\to 0,\ V\to 0)$. The transformed Eqs. (1) and (3) read

$$\begin{aligned}\epsilon\zeta_T(1+\epsilon^2\zeta_X^2)^{3/2}=-\Bigg\{&\frac{1}{2}\Big[(1-\epsilon^2\zeta_X^2)(1+\partial_X U-\zeta_X\partial_Z U\\&-\partial_Z V)+2\zeta_X[\partial_Z U+\epsilon^2(\partial_X V-\zeta_X\partial_Z V)]\Big]^2\\&-\frac{1}{2}(1+\epsilon^2\zeta_X^2)^2-\epsilon\zeta_{XX}(1+\epsilon^2\zeta_X^2)^{1/2}\\&+\epsilon\alpha\zeta(1+\epsilon^2\zeta_X^2)^2\Bigg\},\end{aligned}\tag{9}$$

$$\begin{aligned}(1-2\nu)\partial_Z^2 U+\epsilon^2[&2(1-\nu)(\partial_X-\zeta_X\partial_Z)^2 U\\&+(\partial_X-\zeta_X\partial_Z)\partial_Z V]=0,\end{aligned}\tag{10}$$

$$\begin{aligned}2(1-\nu)\partial_Z^2 V&+(\partial_X-\zeta_X\partial_Z)\partial_Z U\\&+\epsilon^2(1-2\nu)(\partial_X-\zeta_X\partial_Z)^2 V=0,\end{aligned}\tag{11}$$

and the boundary conditions at the interface $(Z=0)$ become

$$\begin{aligned}0=(1-\nu)\partial_Z V+\nu\partial_X U-(1-\nu)\zeta_X\partial_Z U+\epsilon^2[(1-2\nu)\\\times(\zeta_X^2-\zeta_X\partial_X V)+(1-\nu)\zeta_X^2(\partial_Z V+\partial_X U-\zeta_X\partial_Z U)],\end{aligned}\tag{12}$$

$$\begin{aligned}0=[\partial_Z U+\epsilon^2(\partial_X V-\zeta_X\partial_Z V)](1-\epsilon^2\zeta_X^2)\\+2\epsilon^2\zeta_X(\partial_Z V-\partial_X U+\zeta_X\partial_Z U-1).\end{aligned}\tag{13}$$

Equations (9) through (13) constitute our starting point for the asymptotic analysis to be carried out.

III. ASYMPTOTIC ANALYSIS

These equations are singular in Z, i.e., the boundary conditions at infinity, where the implicit assumption $Z=O(1)$ does not hold anymore, can be satisfied only with the trivial solution $\zeta_X=0$. This difficulty usually does not appear with equations that have an internal length scale, in contrast to the elastic equations, which are devoid of such a scale. To overcome it, one introduces an *inner* region, where $Z=O(1)$, and an *outer* region, where $\eta\equiv\epsilon Z=O(1)$ [5], attempting then to match the solutions in their common domain of validity. In the inner region, Eqs. (10)–(13) hold, and solving the bulk equations may be reduced to ordinary differential equations in Z via expansion in powers of ϵ. In the outer region, derivatives with respect to η and X are the same order of magnitude. Therefore, true partial differential equations have to be solved. It is here that the linearity of the Lamé equations becomes important, because it allows one to solve these equations in terms of Fourier transforms. Nontrivial aspects of the calculation then are largely due to the asymptotic matching of real space functions with Fourier transforms.

Denoting field variables in the outer region by small letters, $u(X,\eta,T)=U(X,Z,T)$, $v(X,\eta,T)=V(X,Z,T)$, we obtain the *outer* equations

$$\begin{aligned}(1-2\nu)\partial_\eta^2 u+2(1-\nu)(\partial_X-\epsilon\zeta_X\partial_\eta)^2 u\\+\epsilon(\partial_X-\epsilon\zeta_X\partial_\eta)\partial_\eta v=0,\end{aligned}\tag{14}$$

$$\begin{aligned}(\partial_X-\epsilon\zeta_X\partial_\eta)\partial_\eta u+2(1-\nu)\epsilon\partial_\eta^2 v\\+\epsilon(1-2\nu)(\partial_X-\epsilon\zeta_X\partial_\eta)^2 v=0,\end{aligned}\tag{15}$$

the *inner* ones being given by Eqs. (10) and (11). Expanding the interface position and the fields in powers of ϵ, we find that the expansion is singular in another sense. If it is assumed that $\zeta=O(1)$, nonlinear terms will *not* arise in the interface equation, a fact that can also be inferred from symmetry considerations and power counting. Scalings that lead to a nontrivial result are $\zeta=O(\epsilon^{-1})$, $u=O(1)$, and $v=O(\epsilon^{-1})$. We therefore write

$$\zeta(X,T)=\epsilon^{-1}\zeta_{-1}(X,T)+\zeta_0(X,T)+\epsilon\zeta_1(X,T)+\cdots,\tag{16}$$

implying $\partial_{\tilde{x}}\zeta(X,T)=O(1)$. Next we set

$$V(Z)=\epsilon^{-1}V_{-1}(Z)+V_0(Z)+\epsilon V_1(Z)+\cdots$$
$$=v(\eta)=\epsilon^{-1}v_{-1}(\eta)+v_0(\eta)+\epsilon v_1(\eta)+\cdots, \tag{17}$$

$$U(Z)=U_0(Z)+\epsilon U_1(Z)+\cdots$$
$$=u(\eta)=u_0(\eta)+\epsilon u_1(\eta)+\cdots, \tag{18}$$

where for brevity we have suppressed the X and T dependences. We then obtain as matching conditions (taking into account the ϵ dependence of η),

$$U_0(Z)\to u_0(0) \quad (Z\to\infty), \tag{19}$$

$$V_{-1}(Z)\to v_{-1}(0) \quad (Z\to\infty), \tag{20}$$

$$U_1(Z)\sim Zu_0'(0)+u_1(0) \quad (Z\to\infty), \tag{21}$$

$$V_0(Z)\sim Zv'_{-1}(0)+v_0(0) \quad (Z\to\infty), \tag{22}$$

where the prime denotes a derivative with respect to the argument (η). To the two leading orders, the inner equations are solved by

$$U_0(X,Z,T)=B_0(X,T), \tag{23}$$

$$V_{-1}(X,Z,T)=D_{-1}(X,T), \tag{24}$$

$$U_1(X,Z,T)=ZA_1(X,T)+B_1(X,T), \tag{25}$$

$$V_0(X,Z,T)=ZC_0(X,T)+D_0(X,T). \tag{26}$$

The outer equations need to be solved only at lowest order in ϵ. For brevity, we rename $\zeta_{-1}(X,T)$ back to $\zeta(X,T)$. Confusion should not arise, since we will never need ζ_0 nor ζ_1. It should be kept in mind, however, that the new $\zeta(X,T)$ is $O(1)$, because it is the prefactor of ϵ^{-1}. The outer solution then reads

$$u_0(X,\eta,T)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{a_0(q,T)+b_0(q,T)[\eta+\zeta(X,T)]\}e^{iqX+|q|[\eta+\zeta(X,T)]}dq, \tag{27}$$

$$v_{-1}(X,\eta,T)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{-i\,\mathrm{sgn}(q)\{a_0(q,T)+b_0(q,T)[\eta+\zeta(X,T)]\}+i\frac{3-4\nu}{q}b_0(q,T)\right\}e^{iqX+|q|[\eta+\zeta(X,T)]}dq. \tag{28}$$

Using the boundary conditions (12) and (13), we can express A_1 and C_0 by B_0 and D_{-1}:

$$A_1(X)=\frac{1}{(1-\nu)(1+\zeta_X^2)^2}\{\zeta_X[2(1-\nu)+(1-2\nu)\zeta_X^2+((1-\nu)\zeta_X^2+2-\nu)B_0'(X)]-(1-\nu-\nu\zeta_X^2)D'_{-1}(X)\}, \tag{29}$$

$$C_0(X)=\frac{1}{(1-\nu)(1+\zeta_X^2)^2}\{\zeta_X^2+[-\nu+(1-\nu)\zeta_X^2][B_0'(X)+\zeta_X(X)D'_{-1}(X)]\}, \tag{30}$$

where we have left out the T dependence (which is ''passively'' present everywhere but important only in the final equation—the Lamé equations are always solved at a fixed time).

The matching conditions at leading order give four equations expressing B_0, D_{-1}, A_1, and C_0 by Fourier integrals involving $a_0(q,T)$ and $b_0(q,T)$. For B_0 and D_{-1}, these integrals are essentially shown in Eqs. (27) and (28), we just need to set $\eta=0$ there. The two remaining equations read

$$A_1(X)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{a_0(q)|q|+b_0(q)[1+|q|\zeta(X)]\}e^{iqX+|q|\zeta(X)}dq, \tag{31}$$

$$C_0(X)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{-iqa_0(q)-ib_0(q)[\mathrm{sgn}(q)+q\zeta(X)]+i\,\mathrm{sgn}(q)(3-4\nu)b_0(q)\}e^{iqX+|q|\zeta(X)}dq. \tag{32}$$

So we have six equations altogether, viz., Eqs. (27)–(32), for the six unknown functions B_0, D_{-1}, A_1, C_0, a_0, and b_0 [20]. Their solution determines the elastic fields, which then can be inserted into the interface equation (9). However, what we need is an explicit analytic solution. Its details are technical, so we just give some essential steps. First, we rewrite the equations in terms of Laplace transforms (rather, one-sided Fourier transforms)

$$\hat{a}_0(x)\equiv\frac{1}{2\pi}\int_0^{\infty}a_0(q)e^{iqx}dq, \tag{33}$$

$$\hat{b}_0(x)\equiv\frac{1}{2\pi}\int_0^{\infty}b_0(q)e^{iqx}dq, \tag{34}$$

for example,

$$B_0(X)=\hat{a}_0[X-i\zeta(X)]+\zeta(X)\hat{b}_0[X-i\zeta(X)]+\text{c.c.} \tag{35}$$

Setting $w=X+iZ$,

$$2\hat{a}_0'(w)\equiv m(X,Z)+in(X,Z), \tag{36}$$

$$2\hat{b}_0(w)\equiv o(X,Z)+ip(X,Z), \tag{37}$$

with m, n, o, and p real functions, we can then reduce our problem to just two (real) equations involving complex quantities. These are the Eqs. (29) and (30) with all quantities expressed by $\hat{a}_0'(w)$ and by $\hat{b}_0(w)$, i.e., finally by m, n, o, and p.

Introducing the abbreviations

$$\lambda_1(X)=\frac{\zeta_X[2-2\nu+(1-2\nu)\zeta_X{}^2]}{(1-\nu)(1+\zeta_X{}^2)^2}, \tag{38}$$

$$\lambda_2(X)=\frac{\zeta_X[2-\nu+(1-\nu)\zeta_X{}^2]}{(1-\nu)(1+\zeta_X{}^2)^2}, \tag{39}$$

$$\lambda_3(X)=-\frac{1-\nu-\nu\zeta_X{}^2}{(1-\nu)(1+\zeta_X{}^2)^2}, \tag{40}$$

$$\lambda_4(X)=\frac{\zeta_X{}^2}{(1-\nu)(1+\zeta_X{}^2)^2}, \tag{41}$$

$$\lambda_5(X)=\frac{-\nu+(1-\nu)\zeta_X{}^2}{(1-\nu)(1+\zeta_X{}^2)^2}, \tag{42}$$

$$\lambda_6(X)=\frac{\zeta_X[-\nu+(1-\nu)\zeta_X{}^2)}{(1-\nu)(1+\zeta_X{}^2)^2}, \tag{43}$$

these two equations become

$$\begin{aligned}&[1-\lambda_2(X)\zeta_X+(3-4\nu)\lambda_3(X)]o(X,-\zeta)\\&\quad+[-\lambda_2(X)+\lambda_3(X)\zeta_X]m(X,-\zeta)\\&\qquad=\lambda_1(X)+[\lambda_2(X)-\lambda_3(X)\zeta_X]\zeta o_X(X,-\zeta)+[-1\\&\qquad\quad+\lambda_2(X)\zeta_X+\lambda_3(X)][n(X,-\zeta)+\zeta p_X(X,-\zeta)]\\&\qquad\quad-2(1-2\nu)\lambda_3(X)\zeta_X p(X,-\zeta),\end{aligned} \tag{44}$$

$$\begin{aligned}&[-1-\lambda_5(X)+\lambda_6(X)\zeta_X]m(X,-\zeta)+[-\lambda_5(X)\zeta_X\\&\quad+(3-4\nu)\lambda_6(X)]o(X,-\zeta)\\&\qquad=\lambda_4(X)-[-1-\lambda_5(X)+\lambda_6(X)\zeta_X]\zeta o_X(X,-\zeta)\\&\qquad\quad+2(1-2\nu)[1-\lambda_6(X)\zeta_X]p(X,-\zeta)+[\lambda_5(X)\zeta_X\\&\qquad\quad+\lambda_6(X)][n(X,-\zeta)+\zeta p_X(X,-\zeta)],\end{aligned} \tag{45}$$

where p_X and o_X denote the derivatives of $p(X,-\zeta)$, $o(X,-\zeta)$ with respect to X at fixed ζ. That is, we first take the derivative and only then allow $\zeta=\zeta(X)$.

Since the four functions involved are real and imaginary parts of analytic functions, only two of them are independent. Indeed, we have

$$n(X,-\zeta)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(X'-X)m(X',0)}{(X'-X)^2+\zeta^2}dX', \tag{46}$$

$$p(X,-\zeta)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(X'-X)o(X',0)}{(X'-X)^2+\zeta^2}dX'. \tag{47}$$

Equations (44) and (45), being linear, can be solved formally for $o(X,-\zeta)$ and $m(X,-\zeta)$. This leads to

$$\begin{aligned}o(X,-\zeta)&=r_1(X)[n(X,-\zeta)+\zeta p_X(X,-\zeta)]\\&\quad+r_2(X)p(X,-\zeta)+r_3(X),\end{aligned} \tag{48}$$

$$\begin{aligned}m(X,-\zeta)&=r_4(X)[n(X,-\zeta)+\zeta p_X(X,-\zeta)]\\&\quad+r_5(X)p(X,-\zeta)+r_6(X)\zeta o_X(X,-\zeta)+r_7(X),\end{aligned} \tag{49}$$

where the functions r_1 through r_7 are given by

$$r_1(X)=\frac{1}{1-2\nu}, \tag{50}$$

$$r_2(X)=\frac{2\zeta_X}{(1-2\nu)(1+\zeta_X^2)}, \tag{51}$$

$$r_3(X)=\frac{-\zeta_X}{(1-2\nu)(1+\zeta_X^2)}, \tag{52}$$

$$r_4(X)=0, \tag{53}$$

$$r_5(X)=2\frac{-1+\nu(1+\zeta_X{}^2)}{1+\zeta_X{}^2}, \tag{54}$$

$$r_6(X)=-1, \tag{55}$$

$$r_7(X)=-\frac{\zeta_X{}^2}{1+\zeta_X{}^2}. \tag{56}$$

The computations leading from Eqs. (27) through (32) to Eqs. (48) and (49) are straightforward but heavy. Some intermediate steps [e.g., the simplification of the $r_i(X)$] as well as a verification of the whole calculation at the end have been done using computer algebra (MAPLE). Due to the chosen scalings (16)–(18), Eqs. (48) and (49) constitute an exact reformulation of the outer elastic problem, which is linear in u and v, so higher-order equations have the same form as the leading order in ϵ. To obtain analytic results, we have to employ some approximation. We assume that $\zeta(X)$, the expansion coefficient of ϵ^{-1}, is itself a numerically small quantity. Then expressions, such as Eqs. (46) and (47) become *Hilbert transforms*. Denoting the Hilbert transform of $f(X)$ by

$$\mathcal{H}[f(X)] \equiv \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(X')}{X'-X} dX', \tag{57}$$

where the integral is to be taken as a principal value, as indicated by the bar, we find, as the limit $\zeta \ll 1$ of Eqs. (48) and (49),

$$o(X,0) = -r_1(X)\mathcal{H}[m(X,0)] - r_2(X)\mathcal{H}[o(X,0)] + r_3(X), \tag{58}$$

$$m(X,0) = -r_5(X)\mathcal{H}[o(X,0)] + r_7(X). \tag{59}$$

This set of linear equations can be solved iteratively. Assuming, and this is the second approximation, that $\zeta_X(X)$ is also numerically small, we may truncate the iteration after the first step, taking accurately into account only terms up to order $\zeta_X^2(X)$. Using $\mathcal{H}\{\mathcal{H}[f(X)]\} = -f(X)$, we can solve the truncated iteration to obtain explicit expressions for $o(X,0),\ m(X,0)$ that read

$$o(X,0) = -(1-2\nu)r_3(X) + \mathcal{H}[r_7(X)], \tag{60}$$

$$m(X,0) = -(1-2\nu)r_7(X) - 2(1-\nu)(1-2\nu)\mathcal{H}[r_3(X)]. \tag{61}$$

Substituting $o,\ m,\ n = -\mathcal{H}[m]$, and $p = -\mathcal{H}[o]$ back into the equations for $A_1,\ C_0,\ B_0$, and D_{-1}, inserting these into Eqs. (23)–(26) and then back into Eq. (9), we finally obtain the sought-for interface equation, in which we rename X into x and T into t (they are the same variables anyway):

$$\frac{\zeta_t(x,t)}{(1+\zeta_x^2)^{1/2}} = \frac{1}{2}\left\{1 - \frac{1}{(1+\zeta_x^2)^2}\left[(1-\zeta_x^2)\left(1 + \frac{2}{\pi}\fint_{-\infty}^{\infty} \frac{\zeta_x(x')}{(x'-x)[1+\zeta_x(x')^2]} dx'\right) + \frac{4\zeta_x^2}{1+\zeta_x^2}\right]^2\right\} + \frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}} - \alpha\zeta. \tag{62}$$

The arguments of ζ (x or x' and t) have been suppressed wherever possible without ambiguity.

The nonlocal term drives the instability. It is interesting to see whether Eq. (62) will reproduce a *coarsening dynamics* of the type discussed in Ref. [18]. For in that case whether a groove grows or shrinks does not depend on local quantities (such as the curvature) alone. Hence, the scenario described cannot be expected from a merely local equation.

This is the melt-crystallization case [18]. It is easily checked that linearization of this equation produces the linear dispersion relation (7). The corresponding equation with dynamics controlled by surface diffusion is obtained simply by preceding the right-hand side with the operator $-[(1+\zeta_x^2)^{-1/2}\partial_x]^2$:

$$\zeta_t(x,t) = \partial_x \frac{1}{\sqrt{1+\zeta_x^2}} \partial_x \left(\frac{1}{2}\left\{\frac{1}{(1+\zeta_x^2)^2}\left[(1-\zeta_x^2)\left(1 + \frac{2}{\pi}\fint_{-\infty}^{\infty} \frac{\zeta_x(x')}{(x'-x)[1+\zeta_x(x')^2]} dx'\right) + \frac{4\zeta_x^2}{1+\zeta_x^2}\right]^2\right\} - \frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}} + \alpha\zeta\right). \tag{63}$$

For the system described by this equation, coarsening dynamics does not yet seem to have been studied in the literature, therefore it will be briefly discussed in the following section.

IV. SOME SIMULATION RESULTS

Both Eqs. (62) and (63) have been simulated numerically. Figure 1 gives an example for period-doubling dynamics found in the nonconservative case when an initially periodic sinusoidal structure is submitted to a small perturbation of twice its basic periodicity.

A similar result is obtained for conservative dynamics and the same parameter α, see Fig. 2. In both cases, α was close to 1, hence the instability was weak. For smaller values of α, we found stronger differences between the behavior of the two systems. The main statement that can be made, however, is that in both of them similar coarsening scenarios are operative, since coarsening allows a reduction of the overall elastic energy. Because of the restriction that the average interface height has to remain constant in the conservative case, in the latter coarsening proceeds more slowly, as can be verified by comparison of Figs. 1 and 2.

In Fig. 3, we give the dynamics obtained from a *local* perturbation of a periodic interface. The process of coarsening is more complex than the simple period-doubling scenario suggested in Ref. [18]. Frustration effects come into play when grooves compete via the nonlocal interaction, so it may happen that the next-nearest neighbor of the leading groove is not a winner of the competition but rather the third neighbor.

An extensive overview of the properties of Eqs. (62) and (63) is not intended here. It will be the subject of a different publication [21]. Suffice it to say that we find these amplitude equations to reproduce several qualitative features of the dynamics of the system faithfully, including *nonlocally induced coarsening*. Differences with the full dynamics [18] appear at large amplitudes. For vanishing gravity (i.e., $\alpha = 0$), the amplitude is not predicted to saturate in the full model but does so in the amplitude equation. This is not so surprising given that the latter was derived assuming small

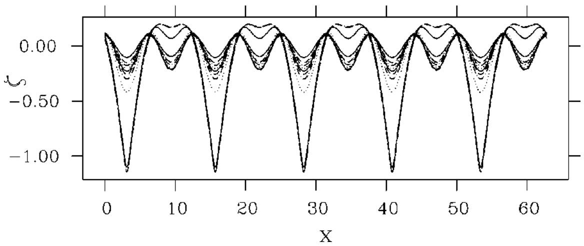

FIG. 1. Period doubling for the interface profile. The initial interface is a cosine with wavelength 2π and amplitude 0.1, with every odd minimum made deeper by 5%. $\alpha=0.97$ and time interval between curves is $\Delta t=10.0$.

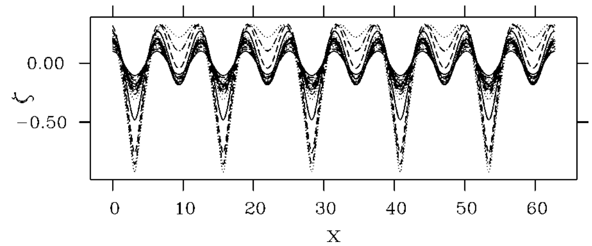

FIG. 2. The initial interface is a cosine with wavelength 2π and amplitude 0.1, with every odd minimum made deeper by 5%. $\alpha=0.97$ and time interval between curves is $\Delta t=10.0$.

amplitudes [more precisely a small prefactor of the $O(\epsilon^{-1})$ term in the amplitude expansion]. In principle, systematic improvement is possible by continuing the iteration scheme truncated after the first step and including higher powers of ζ and ζ_x. In practice, this improvement results in additional technical difficulties.

It is certainly not possible to claim genericity of Eqs. (62) and (63). Their derivation accounts for only the lowest-order (i.e., quadratic) nonlinearities exactly, while resumming some nonlinearities to infinite order, which are presumably specific for the elastic system. However, if there is a universal amplitude equation for the systems considered, then it must contain the terms that our derivation produces exactly, and it may be sufficient to consider just these terms.

V. GENERIC EQUATIONS

Before looking at the generic equation(s) derivable from our particular case, let us briefly recapitulate the general situation concerning amplitude equations of universal validity.

There is an overabundance of nonequilibrium systems that spontaneously build up an organized pattern from an initially structureless state when they are taken sufficiently far from equilibrium [9]. Typical examples are present in hydrodynamical systems (e.g., Rayleigh-Bénard convection), chemical reactions (e.g., Turing systems), crystal growth (the Mullins-Sekerka instability), and so on. Despite the fact that the underlying physical and chemical mechanisms are diverse in these systems, sufficiently close to the instability threshold they all fall within the same universality class. Let q_c denote the wave number of the emerging ordered pattern, and $h(x,t)$ a field describing the pattern (which can stand for a component of the velocity field in hydrodynamics, or an interface position in crystal growth, etc.) along the x direction. Close to the threshold it is known [9] that $h \sim A(x,t)e^{iq_c x}$, where $e^{iq_c x}$ describes the (rapid) periodic variation of the field due to the birth of order, while A is a slowly varying (slow with respect to $e^{iq_c x}$) amplitude. A obeys the following equation:

$$A_t=A+A_{xx}-|A|^2A, \tag{64}$$

where coefficients can always be set to unity by an appropriate rescaling. This equation is universal in the sense that its form depends only on translational and rotational symmetries. It is usually referred to as the *amplitude equation* or the Ginzburg-Landau equation [9]. The amplitude equation describes the nature of the bifurcation from the structureless state as well as instabilities of the ordered pattern with respect to wavelength modulations (the Eckhaus instability [9]).

A description in terms of a slowly varying amplitude makes sense only if A varies in a slow fashion in comparison with the pattern wavelength $1/q_c$. This requirement fails if $q_c \to 0$. In that case a separation between a fast and slow variation is illegitimate. One has thus to resort to a (singular) expansion in the manner of Sivashinsky [4]. Because q_c is small, it provides an appropriate parameter of expansion. This is also known as the *long-wavelength limit*. In that limit the field h obeys generically the Kuramoto-Sivashinsky equation (or the damped form with λ the damping rate), which is written in the canonical form,

$$h_t=-\lambda h-h_{xx}-h_{xxxx}+h_x^2. \tag{65}$$

Again this equation is universal [22,23] in the limit $q_c \to 0$. Because of the smallness of q_c (long wavelength) any nonlocal effect present in the original constitutive equations disappears, and one recovers again a local equation. This is true only because a length scale is present and serves as a cutoff. It is akin to the Debye length that makes the nonlocal response function of the Coulomb gas local. There is, however, a myriad of situations where long-range interactions play a decisive role at all scales, and where it is hardly believable that the above equation (or a similar equation) should arise even in the long-wavelength limit. Typical situations with long-range interactions are electrostatic and magnetic systems, problems with elastic fields, and so on. These systems are devoid of an intrinsic length scale and it is highly desir-

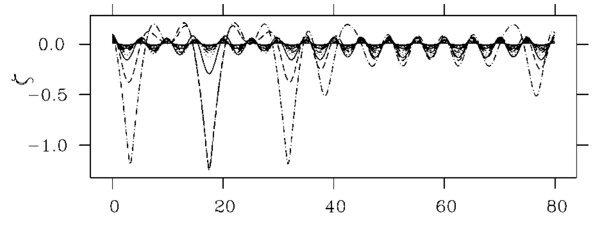

FIG. 3. Coarsening of the interface profile. The fourth groove of the initial interface has been made deeper by 1%. $\alpha=0.90$ and initial wavelength is 2π. Time interval between curves is $\Delta t=10.0$.

able to derive the appropriate generic form of the corresponding evolution equation in the long-wavelength limit. A derivation on the basis of a gradient expansion of an appropriately chosen general nonlinear operator in the spirit of Ref. [22] will be given elsewhere [15].

Here we argue that we can essentially read off the form of that equation from our result (62). To this end, we simply expand to second order in ζ_x and drop all higher-order terms. As noted before, the kept terms are exact to that order. The ''generic'' evolution equation with long-range interaction obtained this way reads

$$\zeta_t = -\alpha\zeta + \zeta_{xx} - \frac{2}{\pi}\fint_{-\infty}^{\infty}\frac{\zeta_x(x')}{x'-x}dx' - 2\zeta_x^2 - \frac{1}{2}\left(\frac{2}{\pi}\fint_{-\infty}^{\infty}\frac{\zeta_x(x')}{x'-x}dx'\right)^2. \tag{66}$$

Comparing this equation with the KS one, Eq. (65), we see that it is not exactly a generalization. Since the nonlocal term drives the instability, the diffusive term need not be negative and hence there is no need for a fourth-order term. Rather, Eq. (66) is *complementary* to the Kuramoto-Sivashinsky result. Of course, the analog of Eq. (66) with a negative sign of ζ_{xx} may also appear and then the dispersion relation would need a different ultraviolet stabilization. If this term were fourth order in q, we would obtain a linear term $\propto \zeta_{xxxx}$ and an equation that is a common generalization of the KS equation and Eq. (66).

Closer inspection shows that this equation can be only a particular case of the generic equation. For, in general, the amplitude equation will have coefficients in front of each of its six terms. Four of these can be made equal to one via division by a common prefactor and rescaling of space, time, and the amplitude ζ itself. So our final equation should contain two independent nondimensional parameters, but it has only one, α. We suspect that this is due to a hidden symmetry, and a natural candidate for this symmetry is the variational nature of the basic evolution dynamics, Eqs. (1)–(3). In the fully generic equation, the additional parameter would be the ratio of the coefficients of the two nonlinear terms. It is therefore easy to conjecture the form of this equation:

$$\zeta_t = -\alpha\zeta + \zeta_{xx} - \frac{2}{\pi}\fint_{-\infty}^{\infty}\frac{\zeta_x(x')}{x'-x}dx' - 2\zeta_x^2 - \frac{\beta}{2}\left(\frac{2}{\pi}\fint_{-\infty}^{\infty}\frac{\zeta_x(x')}{x'-x}dx'\right)^2, \tag{67}$$

where we have introduced a second parameter β. The corresponding universal equation for conservative dynamics would be obtained by preceding the complete right-hand side of this equation by the operator $-\partial_x^2$.

We have simulated Eq. (66) and find that it still describes coarsening, this time without saturation. An example of a simulation is displayed in Fig. 4. We see a single groove or ''finger'' survive after a long time.

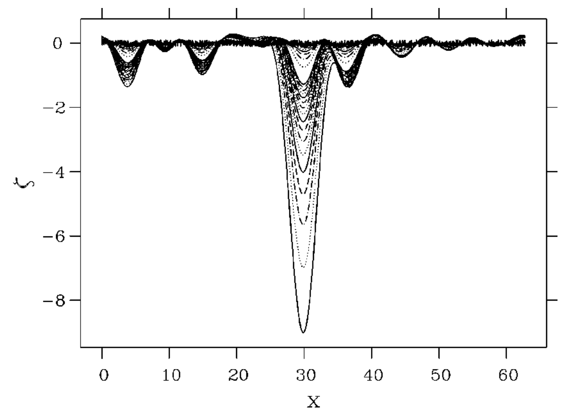

FIG. 4. Evolution according to the generic equation (67). $\alpha = 0.8$, $\beta = 1.0$. The initial interface is a random structure (visible near $z=0$). Time interval between curves is $\Delta t = 1.0$ up to $t = 20.0$, thereafter $\Delta t = 0.1$, and the evolution is shown up to $t = 21.6$. Note the spectacular acceleration of the ''winning groove.''

On physical grounds, a nonsaturating amplitude is to be expected in some situations involving the Grinfeld instability [18]. Nevertheless, even in such a case one should *a priori* not believe that a long-wave equation predicting an ever-increasing amplitude correctly describes the long-time asymptotics, because the interface eventually leaves the domain of validity of the long-wave assumption. This happens when the amplitude becomes larger than the wavelength. Therefore, an equation, such as Eq. (67) should be expected to describe some intermediate-time asymptotics at best.

Things are different for other universal equations such as the Ginzburg-Landau equation (64) and the (damped) Kuramoto-Sivashinsky equation (65), which usually do work in the long-time limit. But this is due to the fact that in these cases the final (average) amplitude does saturate and is predicted to be small. However, if the cubic term in Eq. (64) is positive, which necessitates a fifth-order description to obtain saturation of the amplitude, the final asymptotics cannot ordinarily be trusted either, because then the amplitude may not be small contrary to the original assumptions.

On the other hand, while any amplitude description leading to nonsaturating amplitudes may seem self-defeating in this sense, there appear to be systems, where the domain of validity of the amplitude equation extends beyond that of the assumptions made in its derivation. Recently, a singular long-wave equation was derived for step-flow growth without desorption [24]. It predicts an amplitude growing as the square root of time, i.e., without bounds. Currently available evidence on the basis of numerical solutions of the exact model equations seems to corroborate that prediction. Hence, it is imperative to compare the long-time limit of the amplitude description to whatever information is available about the long-time behavior of the full system, in order to assess its validity.

Equation (67) should be thought of as being the first in a hierarchy of generic equations describing the initial and intermediate stages of evolution of a number of different systems. It is relevant to systems the linear dispersion relation of which is quadratic. For other systems, such as the Saffman-

Taylor system, where the dispersion relation is cubic, we will have a different linear term in the equation (the Laplacian of a Hilbert transform), and it is an interesting question whether the appearing nonlinearities will remain the same. Following the same strategy for the Saffman-Taylor problem, we arrive at the following equation:

$$\zeta_t = -\frac{V}{\pi} \fint_{-\infty}^{\infty} \frac{\zeta_x(x')}{x'-x} dx' - \frac{\gamma B}{\pi} \fint_{-\infty}^{\infty} \frac{\zeta_{xxx}(x')}{x'-x} dx' + V\zeta_x^2 + \gamma B \zeta_x \zeta_{xxx}, \tag{68}$$

where V is the flat-interface velocity, γ the surface tension, and $B = b^2/12\mu$ with b the thickness of the Hele-Shaw cell and μ the fluid viscosity. Of course the linear terms are always quite trivial, but what is noteworthy is that $\beta = 0$, i.e., the nonlinear terms are local, like in the KS equation. (Under the usual assumptions for long-wave equations, the second local nonlinearity is small in comparison with the KS one.) We call this limit the Sivashinsky limit where nonlocality is present in the linear terms only, as was derived originally by Sivashinsky for flame propagation [4]. This contrasts with our Eq. (67) that involves in the long-wavelength limit a *nonlinear and nonlocal* term. Therefore, this equation should introduce a new universality class with long-range interactions.

VI. CONCLUSIONS

To summarize, our method of derivation is applicable whenever the general solution of the bulk equations can be found by a transformation method. Usually this means they must be *linear* (with constant coefficients). A second condition, requiring that the dispersion relation can be transformed to long-wave form, should be always satisfiable. The generalization to three dimensions is straightforward, in principle. Our approach opens up the road to a realm of amplitude equations constituting a hitherto unknown type, the investigation of which seems worthwhile both from the mathematical and physical points of view; a new line of research should be stimulated.

The fact that the inner expansion produces only polynomials in Z (and hence the outer solution is identical to the *uniform* approximation [25]) strongly suggests that the derivation may even be achieved without the detour via a long-wave approach, using a regular perturbation expansion of the outer equations. This would seem to indicate that we are not really dealing with an equation that is valid only in the long-wave limit but rather with an ordinary (but *strongly nonlinear*) amplitude equation valid at all wavelengths for small enough amplitudes, thus *a posteriori* justifying the formal device of rescaling to large wavelengths. On the other hand, the first of our approximations corresponds to setting $|q|\zeta$ equal to zero in the exponents of Eqs. (27) and (28), which, after taking into account all rescalings, means that the amplitude is small compared to the wavelength.

Finally, the amplitude equation is much faster (and easier) to handle numerically than previous schemes. Like the KS equation, for example, its leading behavior described by Eq. (67) should be generic for systems undergoing long-wavelength instabilities with long-range interactions.

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